

ON THE MOTION OF A SELF-GRAVITATING INCOMPRESSIBLE FLUID WITH FREE BOUNDARY AND CONSTANT VORTICITY: AN APPENDIX

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ABSTRACT. In a recent work [1] the authors studied the dynamics of the interface separating a vacuum from an inviscid incompressible fluid, subject to the self-gravitational force and neglecting surface tension, in two space dimensions. The fluid is additionally assumed to be irrotational, and we proved that for data which are size ϵ perturbations of an equilibrium state, the lifespan T of solutions satisfies $T \gtrsim \epsilon^{-2}$. The key to the proof is to find a nonlinear transformation of the unknown function and a coordinate change, such that the equation for the new unknown in the new coordinate system has no quadratic nonlinear terms. For the related irrotational gravity water wave equation with constant gravity the analogous transformation was carried out by the last author in [3]. While our approach is inspired by the last author's work [3], the self-gravity in the present problem is a new nonlinearity which needs separate investigation. Upon completing [1] we learned of the work of Ifrim and Tataru [2] where the gravity water wave equation with constant gravity and *constant vorticity* is studied and a similar estimate on the lifespan of the solution is obtained. In this short note we demonstrate that our transformations in [1] can be easily modified to allow for *nonzero constant vorticity*, and a similar energy method as in [1] gives an estimate $T \gtrsim \epsilon^{-2}$ for the lifespan T of solutions with data which are size ϵ perturbations of the equilibrium. In particular, the effect of the constant vorticity is an extra linear term with constant coefficient in the transformed equation, which can be further transformed away by a bounded linear transformation. This note serves as an appendix to the aforementioned work of the authors.

1. INTRODUCTION

We study the motion of the interface separating a vacuum from an inviscid, incompressible fluid, subject to the self-gravitational force in two spatial dimensions. We assume that the fluid domain is bounded and simply connected, the surface tension is zero, and the initial vorticity of the fluid is a constant. Denoting the fluid domain by $\Omega(t)$, the fluid velocity by \mathbf{v} , and the pressure by P , the evolution is governed by the Euler-Poisson system:

$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P - \nabla \phi & \text{in } \Omega(t), t \geq 0, \\ \operatorname{div} \mathbf{v} = 0, \quad \operatorname{curl} \mathbf{v} = 2\omega_0 \in \mathbb{R} & \text{in } \Omega(t), t \geq 0, \\ P = 0 & \text{on } \partial\Omega(t), \end{cases} \quad (1.1)$$

where the self-gravity Newtonian potential satisfies

$$\begin{cases} \Delta \phi = 2\pi \chi_{\Omega(t)}, \\ \nabla \phi(\mathbf{x}) = \iint_{\Omega(t)} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^2} d\mathbf{y}. \end{cases} \quad (1.2)$$

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Note that applying the curl operator to the first line in equation (1.1) implies that the vorticity $\omega := \operatorname{curl} \mathbf{v}$ satisfies $\omega_t + \mathbf{v} \cdot \nabla \omega = 0$, and therefore if $\mathbf{v}(0)$ has constant vorticity $2\omega_0$, the same holds for $\mathbf{v}(t)$ for all times at which the latter is defined, that is $\operatorname{curl} \mathbf{v} = 2\omega_0$. Similarly, the incompressibility of the flow, $\operatorname{div} \mathbf{v} = 0$, implies that the area of the domain enclosed by the fluid does not change during the evolution. Without loss of generality we fix the normalization $|\Omega| = \pi$. We also assume that $\omega_0^2 < \pi$. As we will show below this will be necessary to ensure the validity of the Taylor sign condition which is necessary for local well-posedness. When the vorticity is assumed to be zero a family of time-independent equilibrium solutions is given by perfect balls moving with constant velocity. In the recent preprint [1] we proved that for data which are a perturbation of size ϵ of these equilibria, the lifespan T of the solution satisfies the lower bound $T \gtrsim \epsilon^{-2}$. In the irrotational case the velocity satisfies $\Delta \mathbf{v} = 0$ in $\Omega(t)$ and the Euler-Poisson system (1.1)-(1.2) can be reduced to a system on the boundary $\partial\Omega(t)$. The key to the estimate above on the lifespan of the solution is then to find a new unknown function and a change of coordinates such that the new unknown satisfies a nonlinear equation in the new coordinates with no quadratic terms in the nonlinearity. For the related irrotational gravity water wave equation with constant gravity the analogous transformation was carried out by the last author in [3]. While our approach is inspired by the last author's work [3], the self-gravity in our problem is a new nonlinearity which requires separate investigation. Upon completion of [1] we learned of the work of Ifrim and Tataru [2], where the gravity water wave equation with constant gravity and *constant vorticity* is studied and a lifespan estimate $T \gtrsim \epsilon^{-2}$ is proved for data which are perturbations of size ϵ of the equilibrium. In this short note we demonstrate that our transformations in [1] can be easily modified to allow for nonzero constant vorticity, and a similar energy method as in [1] gives an estimate $T \gtrsim \epsilon^{-2}$ for the lifespan T of solutions with data which are size ϵ perturbations of the equilibrium.

In the remainder of this note we adopt the notation introduced in [1] and to avoid repetition refer the reader to [1] for the definitions of the symbols we use. Note that the vector field $\mathbf{v}_0 := \omega_0(y, -x)$ satisfies $\operatorname{curl} \mathbf{v}_0 = 2\omega_0$ and $\operatorname{div} \mathbf{v}_0 = 0$, and therefore $\mathbf{v} := \mathbf{v} - \mathbf{v}_0$ is curl and divergence free. Using complex variable notation it follows that $z_t + i\omega_0 z$ is the boundary value of an anti-holomorphic function in Ω , or in other words $(I - \overline{H})(z_t + i\omega_0 z) = 0$, where H denotes the Hilbert transform for $\partial\Omega$. With this observation, and with the notation

$$a := -\frac{1}{|z_\alpha|} \frac{\partial P}{\partial \mathbf{n}},$$

the system (1.1) reduces to the following system on the boundary $\partial\Omega$:

$$\begin{cases} z_{tt} + iaz_\alpha = -\frac{\pi}{2}(I - \overline{H})z \\ H(\overline{z}_t - i\omega_0 \overline{z}) = \overline{z}_t - i\omega_0 \overline{z} \end{cases} \quad (1.3)$$

We refer the reader to [1] for the derivation of (1.3). Note that $z(t, \alpha) = e^{-i\omega_0 t + i\alpha}$ is a solution to (1.3) with $a = \pi - \omega_0^2$. It follows that for the Taylor sign condition $\frac{\partial P}{\partial \mathbf{n}} < 0$ to hold we need to impose the condition $\omega_0^2 < \pi$. The following theorem is the main result of this note.

Theorem 1.1. *Let Ω_0 be a bounded simply-connected domain in \mathbb{C} with smooth boundary $\partial\Omega_0$ satisfying $|\Omega_0| = \pi$, and denote the associated Hilbert transform by H_0 . Suppose $z_0(\alpha) = e^{i\alpha} + \epsilon f(\alpha)$ is a parametrization of $\partial\Omega_0$ and $z_1(\alpha) = v_0 - i\omega_0 e^{i\alpha} + \epsilon g(\alpha)$ where f and g are smooth, $H_0(\overline{z}_1 - i\omega_0 \overline{z}_0) = \overline{z}_1 - i\omega_0 \overline{z}_0$, $v_0 \in \mathbb{C}$ is a constant, and $\omega_0^2 < \pi$. Then there is $T > 0$ and a unique classical solution $z(t, \alpha)$ of (1.3) on $[0, T)$ satisfying $(z(0, \alpha), z_t(0, \alpha)) = (z_0(\alpha), z_1(\alpha))$. Moreover if $\epsilon > 0$ is sufficiently small the solution can be extended at least to $T^* = c\epsilon^{-2}$ where c is a constant independent of ϵ .*

Remark 1.2. *As in [1] the normalization $|\Omega| = \pi$ is for notational convenience and can be removed, and the constant v_0 is to account for the equilibrium solutions described by balls moving at constant velocity.*

2. PROOF OF THEOREM 1.1

In this section we present the proof of Theorem 1.1. The proof follows closely the proof of Theorem 1.1 in [1] and here we only provide the extra computations needed for the argument in [1] to also prove Theorem 1.1 above.

We introduce the notation $\mathfrak{z}(t, \alpha) = e^{i\omega_0 t} z(t, \alpha)$ and note that $H\bar{\mathfrak{z}}_t = \bar{\mathfrak{z}}_t$. Note that since the factor $e^{i\omega_0 t}$ is independent of α one can replace z by \mathfrak{z} in the definition of the Hilbert transform, and in particular the conclusions of Lemma 3.7 in [1] remain valid with z replaced by \mathfrak{z} . The system (1.3) is written in terms of \mathfrak{z} as

$$\begin{cases} \mathfrak{z}_{tt} + ia\mathfrak{z}_\alpha = -\frac{\pi}{2}(I - \bar{H})\mathfrak{z} + 2i\omega_0\mathfrak{z}_t + \omega_0^2\mathfrak{z} \\ H\bar{\mathfrak{z}}_t = \bar{\mathfrak{z}}_t \end{cases}. \quad (2.1)$$

We first show the validity of the Taylor sign condition provided $\omega_0^2 < \pi$, from which local well-posedness follows as in [1]. Let $\tilde{\Omega}(t)$ be the domain with $\partial\tilde{\Omega}(t)$ parametrized by $\mathfrak{z}(t, \cdot)$. We introduce the Riemann mapping $\Phi(t, \cdot) : \tilde{\Omega}(t) \rightarrow \mathbb{D}$, the function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$e^{ih(t, \alpha)} = \Phi(t, \mathfrak{z}(t, \alpha)), \quad (2.2)$$

and the new unknowns in the Riemann mapping coordinates:

$$\begin{aligned} \mathfrak{Z}(t, \alpha') &= \mathfrak{z}(t, h^{-1}(t, \alpha)), & \mathfrak{Z}_t(t, \alpha') &= \mathfrak{z}_t(t, h^{-1}(t, \alpha)), \\ \mathfrak{Z}_{tt}(t, \alpha') &= \mathfrak{z}_{tt}(t, h^{-1}(t, \alpha)), & \mathfrak{Z}_{ttt}(t, \alpha') &= \mathfrak{z}_{ttt}(t, h^{-1}(t, \alpha)). \end{aligned} \quad (2.3)$$

The new unknowns satisfy the following system on the unit circle

$$\begin{cases} \mathfrak{Z}_{tt} + i\mathcal{A}\mathfrak{Z}_{,\alpha'} = -(\pi - \omega_0^2)\mathfrak{Z} + \bar{G}, \\ \mathbb{H}\bar{\mathfrak{Z}}_t = \bar{\mathfrak{Z}}_t. \end{cases} \quad (2.4)$$

where $\mathcal{A} \circ h = ah_\alpha$, \mathbb{H} is the Hilbert transform associated to the unit circle, and G is given by

$$G = \frac{\pi}{2} ((I + H)\bar{\mathfrak{z}}) \circ h^{-1} - 2\omega_0 i \bar{\mathfrak{z}}_t \circ h^{-1}.$$

Note that \bar{G} is the boundary value of an anti-holomorphic function in the unit disc, and therefore a similar argument as the proof of Proposition 7.1 in [1] gives the following result.

Proposition 2.1. $\mathcal{A}_1 := \mathcal{A}|\mathfrak{Z}_{,\alpha'}|^2$ is positive and is given by

$$\begin{aligned} \mathcal{A}_1 &:= \frac{1}{8\pi} \int_0^{2\pi} |\mathfrak{Z}_t(t, \beta') - \mathfrak{Z}_t(t, \alpha')|^2 \csc^2 \left(\frac{\beta' - \alpha'}{2} \right) d\beta' \\ &\quad + \frac{\pi - \omega_0^2}{8\pi} \int_0^{2\pi} |\mathfrak{Z}(t, \beta') - \mathfrak{Z}(t, \alpha')|^2 \csc^2 \left(\frac{\beta' - \alpha'}{2} \right) d\beta' > 0. \end{aligned}$$

We next turn to the proof of long-time existence for small initial data. Note that \mathfrak{z}_t is now the small quantity corresponding to z_t in the irrotational case. The key point for the proof of Theorem 1.1 is that $\delta := (I - H)\varepsilon$, where $\varepsilon := |z|^2 - 1 = |\mathfrak{z}|^2 - 1$, still satisfies an equation without quadratic nonlinear terms. The following proposition is the analogue of Proposition 3.15 in [1].

Proposition 2.2. *Let E be as in Lemma 3.13. in [1]. Then the quantities δ and $\delta_t = \partial_t \delta$ satisfy*

$$\begin{aligned} (\partial_t^2 + ia\partial_\alpha - \pi - 2\omega_0 i\partial_t)\delta = \tilde{\mathcal{N}}_1 := & \frac{\pi}{2}(I - H)E(\varepsilon) + \frac{\pi}{2}[E(\mathfrak{z}), H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} - 2[\mathfrak{z}_t, H\frac{1}{\mathfrak{z}_\alpha} + \overline{H}\frac{1}{\bar{\mathfrak{z}}_\alpha}]\partial_\alpha(\mathfrak{z}_t\bar{\mathfrak{z}}) \\ & - \frac{1}{\pi i} \int_0^{2\pi} \left(\frac{\mathfrak{z}_t(\beta) - \mathfrak{z}_t(\alpha)}{\mathfrak{z}(\beta) - \mathfrak{z}(\alpha)} \right)^2 \varepsilon_\beta(\beta) d\beta, \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} (\partial_t^2 + ia\partial_\alpha - \pi - 2\omega_0 i\partial_t)\delta_t = \tilde{\mathcal{N}}_2 := & -ia_t\partial_\alpha\delta + \frac{\pi}{2} \left((I - H)\partial_t E(\varepsilon) - [\mathfrak{z}_t, H]\frac{\partial_\alpha E(\varepsilon)}{\mathfrak{z}_\alpha} \right) + \frac{\pi}{2}[\partial_t E(\mathfrak{z}), H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} \\ & + \frac{\pi}{2}[E(\mathfrak{z}), H]\partial_t \left(\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} \right) + \frac{\pi}{2}E(\mathfrak{z})[\mathfrak{z}_t, H]\frac{\partial_\alpha \left(\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} \right)}{\mathfrak{z}_\alpha} - \frac{\pi}{2}[\mathfrak{z}_t, H]\frac{\partial_\alpha \left(E(\mathfrak{z})\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} \right)}{\mathfrak{z}_\alpha} \\ & + \frac{2}{\pi i}\partial_t \int_0^{2\pi} \frac{(\mathfrak{z}_t(\alpha) - \mathfrak{z}_t(\beta))\partial_\beta(\mathfrak{z}_t(\beta)\bar{\mathfrak{z}}(\beta))\bar{\mathfrak{z}}(\alpha)\bar{\mathfrak{z}}(\beta) \left(\frac{\varepsilon(\alpha)}{\mathfrak{z}(\alpha)} - \frac{\varepsilon(\beta)}{\mathfrak{z}(\beta)} \right)}{|\mathfrak{z}(\beta) - \mathfrak{z}(\alpha)|^2} d\beta \\ & - \frac{1}{\pi i}\partial_t \int_0^{2\pi} \left(\frac{\mathfrak{z}_t(\beta) - \mathfrak{z}_t(\alpha)}{\mathfrak{z}(\beta) - \mathfrak{z}(\alpha)} \right)^2 \varepsilon_\beta(\beta) d\beta, \end{aligned} \quad (2.6)$$

Remark 2.3. *We note that the vorticity $2\omega_0$ gives the extra linear terms $-2\omega_0 i\partial_t \delta$ in (2.5) and $-2\omega_0 i\partial_t \delta_t$ in (2.6).*

Proof. Using (2.1), we write the equation satisfied by ε as

$$(\partial_t^2 + ia\partial_\alpha)\varepsilon = \frac{\pi}{2}(\mathfrak{z}(I - H)\bar{\mathfrak{z}} - \bar{\mathfrak{z}}(I - \overline{H})\mathfrak{z}) + 2\omega_0 i\partial_t \varepsilon + 2(\bar{\mathfrak{z}}_t \mathfrak{z})_t,$$

which implies

$$(I - H)(\partial_t^2 + ia\partial_\alpha)\varepsilon = \frac{\pi}{2}(I - H)(\mathfrak{z}(I - H)\bar{\mathfrak{z}} - \bar{\mathfrak{z}}(I - \overline{H})\mathfrak{z}) + 2\omega_0 i\partial_t \delta + 2[\mathfrak{z}_t, H]\frac{(\bar{\mathfrak{z}}_t \mathfrak{z})_\alpha}{\mathfrak{z}_\alpha}.$$

Arguing as in Proposition 3.15 of [1] and using Lemma 3.7 of [1] we get

$$\begin{aligned} (\partial_t^2 + ia\partial_\alpha - 2\omega_0 i\partial_t)\delta = & \frac{\pi}{2}(I - H)(\mathfrak{z}(I - H)\bar{\mathfrak{z}} - \bar{\mathfrak{z}}(I - \overline{H})\mathfrak{z}) + \frac{\pi}{2}[(I - \overline{H})\mathfrak{z}, H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} \\ & - 2[\mathfrak{z}_t, H\frac{1}{\mathfrak{z}_\alpha} + \overline{H}\frac{1}{\bar{\mathfrak{z}}_\alpha}]\partial_\alpha(\mathfrak{z}_t\bar{\mathfrak{z}}) - \frac{1}{\pi i} \int_0^{2\pi} \left(\frac{\mathfrak{z}_t(\beta) - \mathfrak{z}_t(\alpha)}{\mathfrak{z}(\beta) - \mathfrak{z}(\alpha)} \right)^2 \varepsilon_\beta(\beta) d\beta. \end{aligned}$$

Exactly the same computation as in the proof of Proposition 3.15 in [1] now shows that

$$\frac{\pi}{2}(I - H)(\mathfrak{z}(I - H)\bar{\mathfrak{z}} - \bar{\mathfrak{z}}(I - \overline{H})\mathfrak{z}) + \frac{\pi}{2}[(I - \overline{H})\mathfrak{z}, H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} = \pi\delta + \frac{\pi}{2}(I - H)E(\varepsilon) + \frac{\pi}{2}[E(\mathfrak{z}), H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha},$$

where E is as in Lemma 3.13 in [1]. Combined with the previous identity this gives equation (2.5), and (2.6) follows from differentiating (2.5) and using Lemma 3.7 in [1]. \square

To show that a_t does not contribute quadratic terms to the nonlinearity in the equation for δ_t we record the following analogue of Lemma 3.16 in [1].

Lemma 2.4. *Let K^* denote the formal adjoint of $K := \operatorname{Re} H = \frac{1}{2}(H + \overline{H})$. Then*

$$(I + K^*)(a_t|\mathfrak{z}_\alpha|) = \operatorname{Re} \left[\frac{-i\mathfrak{z}_\alpha}{|\mathfrak{z}_\alpha|} \left\{ 2[\mathfrak{z}_t, H] \frac{\overline{\mathfrak{z}}_{tt\alpha}}{\mathfrak{z}_\alpha} + 2[\mathfrak{z}_{tt}, H] \frac{\overline{\mathfrak{z}}_{t\alpha}}{\mathfrak{z}_\alpha} - [e^{it}g^a, H] \frac{\overline{\mathfrak{z}}_{t\alpha}}{\mathfrak{z}_\alpha} + 2\omega_0 i[\mathfrak{z}_t, H] \frac{\overline{\mathfrak{z}}_{t\alpha}}{\mathfrak{z}_\alpha} \right. \right. \\ \left. \left. + \frac{1}{\pi i} \int_0^{2\pi} \left(\frac{\mathfrak{z}_t(\beta) - \mathfrak{z}_t(\alpha)}{\mathfrak{z}(\beta) - \mathfrak{z}(\alpha)} \right)^2 \overline{\mathfrak{z}}_{t\beta}(\beta) d\beta + \frac{\pi}{2} ([\mathfrak{z}_t, H] \frac{\partial_\alpha g^h}{\mathfrak{z}_\alpha}) \right\} \right].$$

Proof. The proof is the same as that of Lemma 3.16 in [1]. The only modification is that differentiating (2.1) in the time variable we get the following equation for $\overline{\mathfrak{z}}_t$:

$$\overline{\mathfrak{z}}_{ttt} - ia_t \overline{\mathfrak{z}}_{t\alpha} = ia_t \overline{\mathfrak{z}}_\alpha + \frac{\pi}{2} [\mathfrak{z}_t, H] \frac{\overline{\mathfrak{z}}_\alpha}{\mathfrak{z}_\alpha} + \overline{\mathfrak{z}}_t - 2\omega_0 i \overline{\mathfrak{z}}_{tt}.$$

The only new term compared to the equation for $\overline{\mathfrak{z}}_t$ in Lemma 3.16 in [1] is the last term on the right hand side. For this note that if $F(t, \mathfrak{z})$ is the holomorphic function with boundary value $\overline{\mathfrak{z}}_t$ then

$$\overline{\mathfrak{z}}_{tt} = F_t + \frac{\overline{\mathfrak{z}}_{t\alpha} \mathfrak{z}_t}{\mathfrak{z}_\alpha} = F_t + \frac{\overline{\mathfrak{z}}_{t\alpha} \mathfrak{z}_t}{\mathfrak{z}_\alpha}.$$

Since $\overline{\mathfrak{z}}_t$ and \mathfrak{z} have the same holomorphicity properties as $\overline{\mathfrak{z}}_t$ and z in the irrotational case, the rest of the proof is exactly the same as that of Lemma 3.16 in [1]. \square

We now define the change of coordinates k similarly to Remark 3.21 in [1] and show that in the new coordinate $\alpha' = k(t, \alpha)$ the equations for δ and δ_t have no quadratic nonlinearities. The precise identities are given in the following proposition.

Proposition 2.5. *Suppose $z(t, \cdot)$ is a simple closed curve containing the origin in its simply connected interior for each $t \in I$, where I is some time interval, and let k be as defined in Remark 3.21 in [1], but with z replaced by \mathfrak{z} , that is $(I - H)(\log(\overline{\mathfrak{z}}e^{ik})) = 0$. Then*

$$(I - H)k_t = -i(I - H) \frac{\overline{\mathfrak{z}}_t \varepsilon}{\overline{\mathfrak{z}}} - i[\mathfrak{z}_t, H] \frac{(\log(\overline{\mathfrak{z}}e^{ik}))_\alpha}{\mathfrak{z}_\alpha}, \\ (I - H)k_{tt} = -i(I - H) \frac{\overline{\mathfrak{z}}_{tt} \varepsilon + \overline{\mathfrak{z}}_t \varepsilon_t}{\overline{\mathfrak{z}}} + i(I - H) \frac{\overline{\mathfrak{z}}_t^2 \varepsilon}{\overline{\mathfrak{z}}^2} - i[\mathfrak{z}_t, H] \frac{(\log(\overline{\mathfrak{z}}e^{ik}))_{t\alpha} + ik_{t\alpha}}{\mathfrak{z}_\alpha} + i[\mathfrak{z}_t, H] \frac{1}{\mathfrak{z}_\alpha} \partial_\alpha \left(\frac{\overline{\mathfrak{z}}_t \varepsilon}{\overline{\mathfrak{z}}} \right) \\ - i[\mathfrak{z}_{tt}, H] \frac{(\log(\overline{\mathfrak{z}}e^{ik}))_\alpha}{\mathfrak{z}_\alpha} - \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\mathfrak{z}_t(\beta) - \mathfrak{z}_t(\alpha)}{\mathfrak{z}(\beta) - \mathfrak{z}(\alpha)} \right)^2 (\log(\overline{\mathfrak{z}}e^{ik}))_\beta d\beta, \\ (I - H)(ak_\alpha) = [\mathfrak{z}_t, H] \frac{(\overline{\mathfrak{z}}_t \mathfrak{z})_\alpha}{\mathfrak{z}_\alpha} - [\mathfrak{z}_t, H] \overline{\mathfrak{z}}_t \\ - (I - H) \frac{(\overline{\mathfrak{z}}_{tt} + 2\omega_0 i \overline{\mathfrak{z}}_t) \varepsilon}{\overline{\mathfrak{z}}} + (I - H) \frac{e^{-it} g^h \varepsilon}{\overline{\mathfrak{z}}} + [\mathfrak{z}_{tt} - 2\omega_0 i \mathfrak{z}_t - e^{it} g^a, H] \frac{(\log(\overline{\mathfrak{z}}e^{ik}))_\alpha}{\mathfrak{z}_\alpha}.$$

Moreover if F is a holomorphic function with boundary value $\bar{\mathfrak{z}}e^{ik}$ and satisfies $F(t, 0) \in \mathbb{R}_+$ for all $t \in I$, then with the notation $\mathcal{AV}(f) := \frac{1}{2}[z, H] \frac{f}{z}$,

$$\begin{aligned} \mathcal{AV}(\varepsilon) &= 0, \\ \mathcal{AV}(ak_\alpha) &= -(\pi - \omega_0^2) + \frac{\omega_0}{\pi} \int_0^{2\pi} \frac{\bar{\mathfrak{z}}_t \varepsilon \bar{\mathfrak{z}}_\beta}{|\bar{\mathfrak{z}}|^2} d\beta + \frac{1}{2\pi i} \int_0^{2\pi} \bar{\mathfrak{z}}_t \bar{\mathfrak{z}}_{t\beta} d\beta + \frac{1}{2\pi i} \int_0^{2\pi} \frac{(\bar{\mathfrak{z}}_{tt} - e^{-it} g^h) \varepsilon \bar{\mathfrak{z}}_\beta}{|\bar{\mathfrak{z}}|^2} d\beta, \\ &\quad - \frac{1}{2\pi i} \int_0^{2\pi} \left(\frac{\mathfrak{z}_{tt} - 2i\mathfrak{z}_t - e^{it} g^a}{\mathfrak{z}} \right) \partial_\beta \log F d\beta, \\ \operatorname{Re} \mathcal{AV}(k_t) &= \frac{\operatorname{Re}}{2\pi} \int_0^{2\pi} \frac{\bar{\mathfrak{z}}_t \varepsilon}{|\bar{\mathfrak{z}}|^2} \bar{\mathfrak{z}}_\beta d\beta - \frac{\operatorname{Re}}{2\pi} \int_0^{2\pi} \log F \left(\frac{\mathfrak{z} \bar{\mathfrak{z}}_{t\beta} - \bar{\mathfrak{z}}_t \mathfrak{z}_\beta}{\mathfrak{z}^2} \right) d\beta, \\ \operatorname{Re} \mathcal{AV}(k_{tt}) &= \frac{\operatorname{Im}}{2\pi} \int_0^{2\pi} \left(\frac{\mathfrak{z}_{t\beta} \bar{\mathfrak{z}} - \bar{\mathfrak{z}}_t \mathfrak{z}_\beta}{\mathfrak{z}^2} \right) k_t d\beta + \frac{\operatorname{Re}}{2\pi} \partial_t \int_0^{2\pi} \frac{\bar{\mathfrak{z}}_t \varepsilon}{|\bar{\mathfrak{z}}|^2} \bar{\mathfrak{z}}_\beta d\beta + \frac{\operatorname{Re}}{2\pi} \partial_t \int_0^{2\pi} (\log(\bar{\mathfrak{z}}e^{ik}))_\beta \frac{\mathfrak{z}_t}{\mathfrak{z}} d\beta. \end{aligned}$$

Proof. The proof is the same as those of Propositions 3.18, 3.20, and 5.14 in [1]. Indeed for the identities involving k_t and k_{tt} it suffices to note that \mathfrak{z}_t and \mathfrak{z} have the same holomorphicity as z_t and z as in [1] and the derivation does not rely on the first equations in (1.3) and (2.1). The second identity follows from the same argument as in Proposition 3.18 of [1]. The only difference is that instead of $-\pi z + g^a$, the right hand side of the first equation in (2.1) can be written as $-(\pi - \omega_0^2)\mathfrak{z} + e^{it}g^a + 2i\mathfrak{z}_t$. The computation of the averages follows from similar modifications of the proof of Proposition 3.20 in [1]. \square

Note that the computations for $\mathcal{AV}(ak_\alpha)$ and for the static solution in the introduction show that a is close to $\pi - \omega_0^2$, whereas the “negative Klein-Gordon” term in the equation for δ is still $-\pi\delta$. This can be simply rectified by introducing the new unknown

$$\tilde{\delta} := e^{-\omega_0 it} \delta.$$

With this definition $\tilde{\delta}$ satisfies

$$\begin{aligned} (\partial_t^2 + ia\partial_\alpha - (\pi - \omega_0^2))\tilde{\delta} &= \widetilde{\mathcal{M}}_1 := e^{-\omega_0 it} \tilde{\mathcal{N}}_1, \\ (\partial_t^2 + ia\partial_\alpha - (\pi - \omega_0^2))\tilde{\delta}_t &= \widetilde{\mathcal{M}}_2 := e^{-\omega_0 it} (\tilde{\mathcal{N}}_2 + i\tilde{\mathcal{N}}_1). \end{aligned} \tag{2.7}$$

With the same notation as in [1] and with $\tilde{N}_j = \widetilde{\mathcal{M}}_j \circ k^{-1}$, $j = 1, 2$, we rewrite the equations for $\chi := \tilde{\delta} \circ k^{-1}$ and $v = (\partial_t \tilde{\delta}) \circ k^{-1}$ as

$$\begin{aligned} (\partial_t + b\partial_{\alpha'})^2 \chi + iA\partial_{\alpha'} \chi - (\pi - \omega_0^2) \chi &= \tilde{N}_1, \\ (\partial_t + b\partial_{\alpha'})^2 v + iA\partial_{\alpha'} v - (\pi - \omega_0^2) v &= \tilde{N}_2. \end{aligned} \tag{2.8}$$

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Since equation (2.8) has the same form as the equation studied in [1] the energy estimates are exactly the same. In view of Propositions 2.2 and 2.5 and Lemma 2.4 the right hand sides of the equations in (2.8) contain no quadratic terms. Similarly, the identity for k_{tt} in Proposition 2.5 shows that the contributions of $(\partial_t + b\partial_{\alpha'})b$ which arise in the higher energy estimates as in Proposition 5.15 of [1] do not contain quadratic terms. Therefore the only remaining step in the proof is to verify that $\tilde{v} := (I - H)v$ also satisfies an equation with no quadratic nonlinearities, analogous to the equation derived in Proposition

5.11 in [1]. The computation here is similar and we only present the necessary modifications. We use the same proof as in Proposition 5.11 in [1] replacing z by \mathfrak{z} throughout. In the first step in the commutator $[\partial_t^2 + ia\partial_\alpha - (\pi - \omega_0^2), H]\tilde{\delta}_t$ we get the extra term $2\omega_0 i[\mathfrak{z}_t, H]\frac{\tilde{\delta}_{t\alpha}}{\mathfrak{z}_\alpha}$ compared to [1]. This can be written as

$$[\mathfrak{z}_t, H]\frac{\tilde{\delta}_{t\alpha}}{\mathfrak{z}_\alpha} = -\omega_0 i[\mathfrak{z}_t, H]\frac{\tilde{\delta}_\alpha}{\mathfrak{z}_\alpha} + [\mathfrak{z}_t, H]\frac{e^{-\omega_0 it}\partial_\alpha(I-H)\varepsilon_t}{\mathfrak{z}_\alpha} - [\mathfrak{z}_t, H]\frac{e^{-\omega_0 it}[\mathfrak{z}_t, H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha}}{\mathfrak{z}_\alpha}.$$

The last term is already cubic. To see that $[\mathfrak{z}_t, H]\frac{\tilde{\delta}_\alpha}{\mathfrak{z}_\alpha}$ is also cubic note that

$$e^{\omega_0 it}\tilde{\delta} = (I-H)\varepsilon = (I+\overline{H})\varepsilon - (H+\overline{H})\varepsilon = (I+\overline{H})\varepsilon - \mathfrak{z}[\varepsilon, H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} + E(\varepsilon).$$

The contribution of $-\mathfrak{z}[\varepsilon, H]\frac{\varepsilon_\alpha}{\mathfrak{z}_\alpha} + E(\varepsilon)$ to $[\mathfrak{z}_t, H]\frac{\tilde{\delta}_\alpha}{\mathfrak{z}_\alpha}$ is clearly cubic and for $(I+\overline{H})\varepsilon$ note that

$$[\mathfrak{z}_t, H]\frac{e^{-\omega_0 it}\partial_\alpha(I+\overline{H})\varepsilon}{\mathfrak{z}_\alpha} = [\mathfrak{z}_t, H]\frac{1}{\mathfrak{z}_\alpha} + \overline{H}\frac{1}{\mathfrak{z}_\alpha}]e^{-\omega_0 it}\partial_\alpha(I+\overline{H})\varepsilon,$$

which is cubic. The contribution of $[\mathfrak{z}_t, H]\frac{e^{-\omega_0 it}\partial_\alpha(I-H)\varepsilon_t}{\mathfrak{z}_\alpha}$ is shown to be cubic in a similar way. The only other computation which is different from the proof of Proposition 5.11 in [1] is that of the term $II := -2[\mathfrak{z}_t, H]\frac{\partial_\alpha(ia\partial_\alpha\tilde{\delta})}{\mathfrak{z}_\alpha}$ on page 46 of [1], where we use equation (2.1) for \mathfrak{z} instead of the equation for z in [1]. Here the extra terms we get are

$$-2\omega_0 i[\mathfrak{z}_t, H]\frac{\partial_\alpha}{\mathfrak{z}_\alpha}\left(\frac{\mathfrak{z}_t\tilde{\delta}_\alpha}{\mathfrak{z}_\alpha}\right) - \omega_0^2[\mathfrak{z}_t, H]\frac{\partial_\alpha}{\mathfrak{z}_\alpha}\left(\frac{\mathfrak{z}\tilde{\delta}_\alpha}{\mathfrak{z}_\alpha}\right).$$

The first term is already cubic and the second term is identical, up to a multiplicative constant, to one of the terms already computed in the calculation of II in [1]. This shows that the equation for \tilde{v} contains no quadratic nonlinearities and this completes the proof of Theorem 1.1. \square

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